FIITJEE
Solutions to PRMO-2017

1. How many positive integers less than 1000 have the property that the sum of the digits of each such number is divisible by 7 and the number itself is divisible by 3?

Sol. Let the number be abc
$a + b + c = 7k$ (divisible by 7)
number is divisible by 3 i.e. $a + b + c = 3m$
$\Rightarrow a + b + c$ is divisible by 21.
$\Rightarrow 0 \le a \le 9,\ 0 \le b \le 9, 0 \le c \le 9$
$\Rightarrow 0 \le a + b + c \le 27$
$\Rightarrow a + b + c = 21$

Possible values of (a, b, c)			Permutation
9	9	3	$\frac{\lfloor 3}{\lfloor 2} = 3$
9	8	4	$\lfloor 3 = 6$
9	7	5	$\lfloor 3 = 6$
9	6	6	$\frac{\lfloor 3}{\lfloor 2} = 3$
8	8	5	$\frac{\lfloor 3}{\lfloor 2} = 3$
8	7	6	$\lfloor 3 = 6$
7	7	7	$\frac{\lfloor 3}{\lfloor 3} = 1$

Total Permutations = 28
Alternatively,
$^{23}C_2 - 3 \times {}^{13}C_2 + 9 = 28$

2. Suppose a, b are positive real numbers such that $a\sqrt{a} + b\sqrt{b} = 183.\ a\sqrt{b} + b\sqrt{a} = 182.$
Find $\frac{9}{5}$ (a + b).

Sol. $a^{3/2} + b^{3/2} = 183$ …(i)
$a^{1/2}\, b^{1/2}\, (a^{1/2} + b^{1/2}) = 182$ …(ii)
$(a^{1/2} + b^{1/2})^3 = a^{3/2} + b^{3/2} + 3a^{1/2}\, b^{1/2}\, (a^{1/2} + b^{1/2})$
$= 183 + 3 \times 182$
$= 729 = 9^3$
$a^{1/2} + b^{1/2} = 9$
$a^{1/2}\, b^{1/2} = 182/9$
add (i) and (ii)
$(a+b)\ \left(\sqrt{a} + \sqrt{b}\right) = 365$
$\therefore a + b = \frac{365}{9}$
$\therefore \frac{9}{5}(a + b) = 73$

3. A contractor has two teams of workers: team A an team B. Team A can complete a job in 12 days and team B can do the same job in 36 days. Team A starts working on the job and team B joins team A after four days. The team A withdraws after two more days. For how many more days should team B work to complete the job?

Sol. Let total work is W units and A team does a unit work per day and B team does b unit work per day.

$12a = W \Rightarrow a = \frac{W}{12}$ …(i)

$36\ b = W \Rightarrow b = \frac{W}{36}$ …(ii)

let n days more are needed to complete remaining work

$6a + (n+2)\ b = W$

$6\frac{W}{12} + (n+2)\frac{W}{36} = W \Rightarrow n = 16$

4. Let a, b be integers such that all the roots of the equation $(x^2 + ax + 20)(x^2 + 17x + b) = 0$ are negative integers. What is the smallest possible value of a + b ?

Sol. Roots are –ve hence a is +ve and b is +ve integer

$x^2 + ax + 20 = 0 \Rightarrow a$ = sum of factors of 20

$20 = 1 \times 20 = 2 \times 10 = 4 \times 5$

possible values of a are 21, 12 and 9.

$x^2 + 17x + b = 0$ if $17 = p + q$ then $b = pq$

$17 = 1 + 16 = 2 + 15 = 3 + 14 = 4 + 13 = 5 + 12 = 6 + 11 = 7 + 10 = 8 + 9$

possible vales of p are 16, 30, 42, 52, 60, 66 70, 72.

minimum value of a + b = 9 + 16 = 25

5. Let u, v, w be real numbers in geometric progression such that u > v > w. Suppose $u^{40} = v^u = w^{100}$. Find the value of n.

Sol. $v^2 = wu$

$v^n = u^{40} \Rightarrow v^{3n} = u^{120}$

$v^n = w^{60} \Rightarrow v^{2n} = w^{120}$

$v^{3n}\,v^{2n} = u^{120}\,w^{120}$

$v^{5n} = (v^2)^{120}$

$5n = 240 \Rightarrow n = 48$

6. Let the sum $\sum_{n=1}^{9}\frac{1}{n(n+1)(n+2)}$ written in its lowest terms be $\frac{p}{q}$. Find the value of q – p.

Sol. $S_9 = \sum_{n=1}^{9}\frac{1}{n(n+1)(n+2)}$

Let $T_n = \frac{1}{n(n+1)(n+2)}$

$T_n = \frac{1}{2}\left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right]$

$S_9 = T_1 + T_2 + T_3....T_9 = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{10\times 11}\right]$

$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{110}\right] = \frac{1}{2}\left[\frac{55-1}{110}\right] = \frac{27}{110}$

$S = \frac{p}{q} = \frac{27}{110}$

$q - p = 110 - 27 = 83$

7. Find the number of positive integers n, such that $\sqrt{n}+\sqrt{n+1}<11$.

Sol. $\sqrt{n}+\sqrt{n+1}<11$

$\sqrt{n+1}<11-\sqrt{n}$

$n+1<121-n-22\sqrt{n}$

$22\sqrt{n}<120$

$\sqrt{n}<\frac{60}{11}$

$n<\frac{3600}{121}$

$n<29.75$

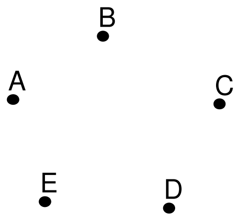
n = 29

8. A pen costs Rs. 11 and a notebook costs Rs. 13. Find the number of ways in which a person can spend exactly Rs. 1000 to buy pens and notebooks.

Sol. let x pens and y notebooks
$11x + 13y = 1000$
y = 5, 16, 27, 38, 49, 60, 71
Hence maximum 7 ways

9. There are five cities A, B, C, D, E on a certain island. Each city is connected to every other city by road. In how many ways can a person starting from city A come back to A after visiting some cities without visiting a city more than once and without taking the same road more than once ? (The order in which he visits the cities also matters: e.g., the routes $A \to B \to C \to A$ and $A \to C \to B \to A$ are different.)

Sol. Let cities be A, B, C and D



Number of ways/routes of the type $(A \to B \to C \to A)$ = ${}^4C_2 \times 2! = 12$ ways
i.e. selecting 2 cities of out of B, C, D, E and permuting them.

Number of ways /routes of the type $(A \to B \to C \to D \to A)$ = ${}^4C_3 \times 3! = 24$ ways
Number of ways /routes of the type $(A \to B \to C \to D \to A)$ = ${}^4C_4 \times 4! = 24$ ways
Total = 60 ways

10. There are eight rooms on the first floor of a hotel, with four rooms on each side of the corridor, symmetrically situated (that is each room is exactly opposite to one other room). Four guests have to be accommodated in four of the eight rooms (that is one in each) such that no two guests are in adjacent rooms or in opposite rooms. In how many ways can the guests be accommodated ?

Sol. Total number of ways
= $\lfloor 4 + \lfloor 4$
= 24 + 24
= 48

11. Let f(x) = sin $\frac{x}{3}$ + cos $\frac{3x}{10}$ for all real x. Find the least natural number n such that f(nπ + x) = f(x) for all real x.

Sol. f(x) = $\sin\frac{x}{3}+\cos\frac{3x}{10}$

f(nπ + x) = sin $\left(\frac{n\pi}{3}+\frac{x}{3}\right)+\cos\left(\frac{3n\pi}{10}+\frac{x}{10}\right)$

= $\frac{n\pi}{3}$ and $\frac{3n\pi}{10}$ should be multiple of 2π

$\Rightarrow \frac{n}{3}$ and $\frac{3n}{10}$ should be even

n should be multiple 6 & 20 both

Hence L.C.M of 6 & 20 will be 60

12. In a class, the total numbers of boys and girls are in the ratio 4 : 3. On one day it was fond that 8 boys and 14 girls were absent from the class and that the number of boys was the square of the number of girls. What is the total number of students in the class ?

Sol. Let number of boys = 4x and number of girls = 3x

Given $4x - 8 = (3x - 14)^2$

$\Rightarrow 4x - 8 = 9x^2 - 84x + 196$

$\Rightarrow 9x^2 - 88x + 204 = 0$

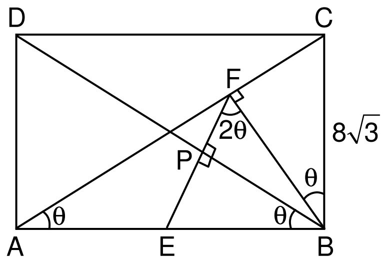
$\Rightarrow (x - 6)(9x - 34) = 0$

x = 6, $x = \frac{34}{9}$ (not possible)

$\therefore$ Total number of student = 7x = 7 × 6 = 42

13. In a rectangle ABCD, E is the midpoint of AB; F is a point on AC such that BF is perpendicular to AC; and FE perpendicular to BD. Suppose BC = $8\sqrt{3}$. Find AB.

Sol. Let AB = a



$FB = 8\sqrt{3}\cos\theta$

BP = FB sin 2θ = $8\sqrt{3}\cos\theta\sin 2\theta = \frac{a}{2}\cos\theta$

$a = 16\sqrt{3}\sin 2\theta$ ..... (1)

$\tan\theta = \frac{8\sqrt{3}}{a}$ ..... (2)

$$a = 16\sqrt{3}\ \frac{2\tan\theta}{1+\tan^2\theta} = \frac{\left(16\sqrt{3}\right)\times 2\times\frac{8\sqrt{3}}{a}}{1+\frac{192}{a^2}} \Rightarrow a = 24$$

14. Suppose x is a positive real number such that {x}, {x} and x are in a geometric progression. Find the least positive integer n such that $x^n > 100$. [Here [x] denotes the integer part of x and {x} = x – [x].)

Sol. Given {x}, [x], x are in G.P.

$\therefore [x]^2 = x\{x\} = x(x - [x])$

$\Rightarrow x^2 - x[x] - [x]^2 = 0$

$$x = \frac{[x]\pm\sqrt{[x]^2+4[x]^2}}{2}$$

$$x = [x]\left(\frac{1+\sqrt{5}}{2}\right)$$

$$[x]+\{x\} = [x]\left(\frac{1+\sqrt{5}}{2}\right)$$

$$\{x\} = [x]\left(\frac{\sqrt{5}-1}{2}\right)$$

$$0 \le [x]\frac{\left(\sqrt{5}-1\right)}{2} < 1$$

$$0 \le [x] < \frac{2}{\sqrt{5}-1}$$

$0 < [x] < \dfrac{\sqrt{5}+1}{2}$ as {x}, [x], x are in G.P. [x] cannot be zero

[x] = 1

$\therefore$ x = [x] + {x} = $1+\dfrac{\sqrt{5}-1}{2} = \dfrac{\sqrt{5}+1}{2}$

$$\left(\frac{\sqrt{5}+1}{2}\right)^n > 100$$

$\therefore$ n = 10

15. Integers 1, 2, 3, ….., n, where n > 2, are written on a board. Two numbers m, k such that 1 < m < n, 1 < k < n are removed and the average of the remaining numbers is found to be 17. What is the maximum sum of the two removed numbers ?

Sol. $\text{Min. A.M.} = \dfrac{\frac{n(n+1)}{2}-(2n-3)}{n-2} = \dfrac{n^2-3n+6}{2(n-2)}$

$\text{Max. A.M.} = \dfrac{\frac{n(n+1)}{2}-5}{n-2} = \dfrac{n^2+n-10}{2(n-2)}$

Now $\dfrac{n^2-3n+6}{2(n-2)} \le 17 \le \dfrac{n^2+n-10}{2(n-2)}$

$\Rightarrow 31 \le n < 35$

Now $\dfrac{\frac{n(n+1)}{2}-(m+k)}{n-2} = 17$

$\Rightarrow$ m + k = $\dfrac{n(n+1)}{2} - 17(n-2)$

$(m + k)_{max}$ occurs at n = 34

$\Rightarrow (m + k)_{max} = 51$

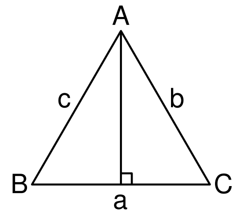
16. Five distinct 2-digit numbers are in a geometric progression. Find the middle term.

Sol. Only possible five distinct numbers for a = 16, $r = \dfrac{3}{2}$

$\therefore$ Numbers are 16, 24, 36, 54, 81

$\therefore$ Middle term = 36

17. Suppose the altitudes of a triangle are 10, 12 and 15. What is the its semi-perimeter?



Sol.. Area of $\Delta ABC = \frac{1}{2}a\times10 = \frac{1}{2}\times b\times12 = \frac{1}{2}c\times15$

$\Rightarrow$ 10a = 12b = 15c

$\Rightarrow \frac{a}{6} = \frac{b}{5} = \frac{c}{4} = k$ (say)

$\Rightarrow$ a = 6k, b = 5k, c = 4k

Also area of $\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\frac{15k}{2}\frac{3k}{2}\frac{5k}{2}\frac{7k}{2}} = \frac{15k^2}{2^2}\sqrt{7}$

$\therefore \frac{15k^2}{4}\sqrt{7} = \frac{1}{2}\times6k\times10 = 30k$

$\Rightarrow k = \frac{30\times4}{15\sqrt{7}} = \frac{8}{\sqrt{7}}$

$\therefore$ Semi-perimeter = $\frac{15k}{2} = \frac{15}{2}\times\frac{8}{\sqrt{7}} = \frac{60}{\sqrt{7}}$

Note : Not possible to write the answer in integer format. Hence bonus marks be awarded.

18. If the real numbers x, y, z are such that $x^2 + 4y^2 + 16z^2 = 48$ and xy + 4yz + 2zx = 24. What is the value of $x^2 + y^2 + z^2$?

Sol. $x^2 + 4y^2 + 16z^2 - 2xy - 8yz - 8yz - 4zx = 48 - 48 = 0$

$\Rightarrow (x - 2y)^2 + (2y - 4z)^2 + (4z - x)^2 = 0$

$\Rightarrow$ x = 2y = 4z = k (say)

$x^2 + 4y^2 + 16z^2 = 48$

$\therefore k^2 + k^2 + k^2 = 48$

$k^2 = 16$

Hence, $x^2 + y^2 + z^2 = k^2 + \frac{k^2}{4} + \frac{k^2}{16} = \frac{21k^2}{16}$

$x^2 + y^2 + z^2 = 21$

19. Suppose 1, 2, 3 are the roots of the equation $x^4 + ax^2 + bx = c$. Find the value of c.

Sol. Fourth root of equation is –6, since sum of roots is zero

$c = -1\cdot2\cdot3\cdot(-6) = 36$

20. What is the number of triples (a, b, c) of positive integers such that (i) a < b < c < 10 and (ii) a, b, c, 10 from the sides of a quadrilateral ?

Sol. 0 < a < b < c < 10

Also a + b + c > 10

Total possible case = $^9C_3 = 84$

But a + b + c = 6, 7, 8, 9, 10 not possible

a + b + c = 6	one way	(1, 2, 3)
a + b + c = 7	one way	(1, 2, 4)
a + b + c = 8	two ways	(1, 2, 5) and (1, 3, 4)
a + b + c = 9	three ways	(1, 2, 6) (1, 3, 5) (2, 3, 4)
a + b + c = 10	four ways	(1, 2, 7) (1, 3, 6) (1, 4, 5) and (2, 3, 5)

Hence total ways = 84 – 11 = 73

21. Find the number of ordered triples (a, b, c) of positive integers such that abc = 108.

Sol. abc = 8 = $3^3 \times 2^2$

This is similar to distribution of identical coins among 3 beggars. Here, we have to distribute powers of 3 among 3 terms x, y & z.

$x + y + z = 3$
Number of ways = ${}^{3+2}C_2 = {}^{5}C_2 = 10$
Similarly, we distributed powers of 2 among three terms x,y & z
$x + y + z = 2$
Number of ways = ${}^{2+3-1}C_2 = {}^{4}C_2 = 6$
Hence, total ways / total number of triplets = $10 \times 6 = 60$ possible triplets.

22. Suppose in the plane 10 pairwise nonparallel lines intersect one another. What is the maximum possible number of polygons (with finite areas) that can be formed ?

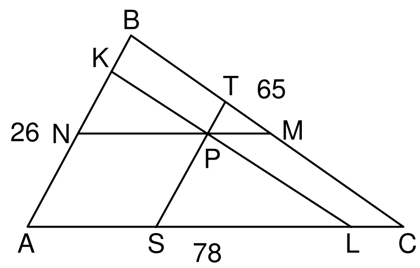
Sol. Maximum number of polygon = ${}^{10}C_3 + {}^{10}C_4 + ... + {}^{10}C_{10}$
$= 2^{10} - ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2)$
$= 2^{10} - (1 + 10 + 45)$
$= 1024 - 56$
= 968 polygons
Note : The answer must be 2 digit integer, hence bonus marks.

23. Suppose an integer x, a natural number n and a prime number p satisfy the equation $7x^2 - 44x + 12 = p^2$. Find the largest value of p.

Sol. $7x^2 - 44x + 12 = p^n$
$(7x - 2)(x - 6) = p^n$
Case-I: Either $x - 6 = 1$ and $7x - 2 = p^n$
$x = 7$
Then $7x - 2 = 7 \times 7 - 2 = 47 = p^n$
$\therefore p = 47$
Case-II: $x - 6 = p^n$ and $7x - 2 = 1$
But $x = \frac{3}{7}$
$x - 6 \neq p^n$
Hence not possible p = 47

24. Let P be an interior point of a triangle ABC whose side length s are 26, 65, 78. The line through P parallel to BC meets AB in K and AC in L. The line through P parallel to CA meets BC in M and BA in N. The line through P parallel to AB meets CA in S and CB in T. If KL, MN, ST are of equal lengths, find this common length.

Sol. **Method-I**



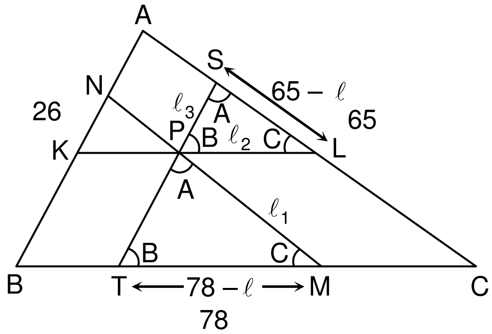
Let $MN = KL = ST = \ell$
$\Delta ALK \sim \Delta ACB$
So, $AL = \frac{6\ell}{5}$, $AK = \frac{2\ell}{5}$
$\Delta CST \sim \Delta CAB$
So, $CS = 3\ell$, $CT = \frac{5\ell}{2}$
$\Delta BNM \sim \Delta BAC$
So, $BN = \frac{\ell}{3}$, $BM = \frac{5\ell}{6}$
Now, $SL = AL + SC - 78 = \frac{21\ell}{5} - 78$
$AS = AL - SL = 78 - 3\ell$
$CL = SC - SL = 78 - \frac{6\ell}{5}$
$\because AS = NP$ and $LC = PM$
So $AS + LC = NP + PM$

$\Rightarrow (78-3\ell)+\left(78-\frac{6\ell}{5}\right)=\ell$

$\Rightarrow \ell = 30$

Which is not possible as point P lies inside the triangle, and possible only when point P lies outside the circle

Method-II



Let PM = ℓ_1, PL = ℓ_2 and PS = ℓ_3

SL = 65 – ℓ and TM = 78 – ℓ

$\angle$TPM = $\angle$NPS = $\angle$A

Similarly $\angle$SPL = B also $\angle$PLS = $\angle$C

and we can find other angles as shown in figure

Now apply Sine Rule in ΔPTM

$\Rightarrow \ell_3 = \ell - \frac{(78-\ell)\sin C}{\sin A}$

and apply Sine Rule in ΔPSL

$\Rightarrow \ell_3 = \frac{\sin C(65-\ell)}{\sin B}$

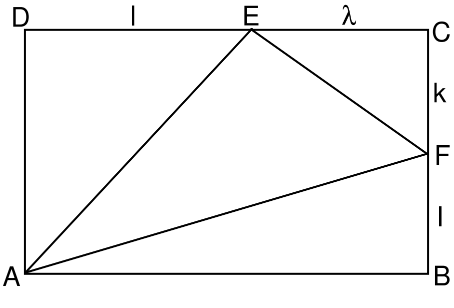
Now again by using Sine Rule $\frac{\sin A}{78}=\frac{\sin B}{65}=\frac{\sin C}{26}$ and eliminating ℓ_3

$\ell = 30$

Which is not possible as point P lies inside the triangle, and possible only when point P lies outside the circle

25. Let ABCD be a rectangle and let E and F be point son CD and BC respectively such that are (ADE) = 16, area (CEF) = 9 and area (ABF) = 25. What is the area of triangle AEF ?

Sol.



ar (ΔADE) = 16

ar (ΔCEF) = 9

ar (ΔABF) = 25

Let Δ be the area of rectangle ABCD

Let $\frac{CE}{ED}=\frac{\lambda}{1}$ and $\frac{CF}{FB}=\frac{k}{1}$

$$\frac{ar(\Delta CEF)}{ar(\Delta CDB)} = \frac{\frac{1}{2}\times CE\times CF}{\frac{1}{2}CD\times CB} = \left(\frac{\lambda}{\lambda+1}\right)\times\left(\frac{k}{k+1}\right)$$

$$\Rightarrow\ ar(\Delta CEF) = \frac{\lambda k}{(\lambda+1)(k+1)}\times\frac{\Delta}{2} = 9 \qquad \ldots(i)$$

$$\frac{ar(\Delta ADE)}{ar(\Delta ADC)} = \left(\frac{1}{1+\lambda}\right) \Rightarrow ar(\Delta ADE) = \frac{\frac{\Delta}{2}}{\lambda+1}$$

$$\Rightarrow \left(\frac{1}{\lambda+1}\right)\left(\frac{\Delta}{2}\right) = 16 \qquad \ldots(ii)$$

$$\frac{ar(\Delta ABF)}{ar(\Delta ABC)} = \left(\frac{1}{1+k}\right) \Rightarrow ar(\Delta ABF) = \left(\frac{1}{1+k}\right)\frac{\Delta}{2}$$

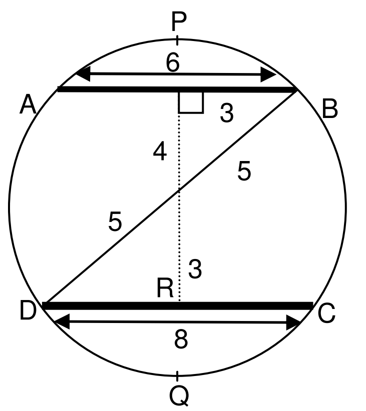
$$\Rightarrow \left(\frac{1}{1+k}\right)\left(\frac{\Delta}{2}\right) = 25 \qquad \ldots(iii)$$

solving (i), (ii) & (iii), we get $\Delta = 80,\ \lambda = \frac{3}{2}$ and $k = \frac{3}{5}$

Thus, area (ΔAEF) = area of rectangle ABCD – [ar(ΔADE) + ar(ΔCEF) + ar(ΔABF)]
= 80 – (16 + 9 + 25)
ar(ΔAEF) = 30 sq. units

26. Let AB and CD be two parallel chords in a circle with radius 5 such that the centre O lies between these chords. Suppose AB = 6, CD = 8. Suppose further that the area of the part of the circle lying between the chords AB and CD is ($m\pi$ + n) / k, where m, n, k are positive integers with gcd (m, n, k) = 1. What is the value of m + n + k ?

Sol.



Radius = 5 units

$\angle$BOT = 37° $\Rightarrow$ $\angle$AOB = 74°

$\angle$ROD = 53° $\Rightarrow$ $\angle$DOC = 106°

area (segment APB) = $\frac{r^2}{2}\theta - ar(\Delta OAB)$

$$= \frac{(5)^2}{2}\left(\frac{74\pi}{180}\right) - \left(\frac{1}{2}\times 6\times 4\right)$$

area (segment DQC) = ar(sector ODQC) – ar(ΔODC)

$$\Rightarrow \frac{r^2}{2}\left(\frac{106}{180}\pi\right)-\frac{1}{2}(8)(3)$$

$\therefore$ area (DCBA) = Area of circle – ar(segment APB) – ar(segment DQC)

$$= \pi(5)^2-\left(\frac{5^2}{2}\left(\frac{74\pi}{180}\right)-12\right)-\left(\frac{25}{2}\left(\frac{106\pi}{180}\right)-12\right)$$

$$= 25\pi-\left(\frac{25}{2}\left(\frac{74+106}{180}\right)\pi\right)+24$$

$$=25\pi-\frac{25\pi}{2}+24$$

$$=\frac{25\pi}{2}+24$$

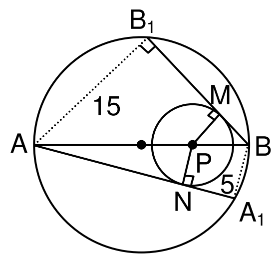
$$=\frac{25\pi+48}{2}$$

$$\frac{m\pi+n}{k}$$

m+n+k = 25 + 48 + 2 = 75

27. Let Ω_1 be a circle with centre O and let AB be a diameter of Ω_1. Let P be a point on the segment OB different from O. Suppose another circle is Ω_2 with centre P lies in the interior of Ω_1. Tangents are drawn from A and B to the circle Ω_2 intersecting Ω_1 again at A_1 and B_1 respectively such that A_1 and B_1 are on the opposite sides of AB. Given that A_1B =5, AB_1 = 15 and OP = 10, find the radius of is Ω_1.

Sol. Let the radius of bigger circle = R
radius of smaller circle = r



$\Delta APN \sim \Delta ABA_1$ $\quad \frac{r}{5}=\frac{R+10}{2R}$

$\Delta BPM \sim \Delta BAB_1$ $\quad \frac{r}{15}=\frac{R-10}{2R}$

Dividing equation (i) by (ii)

we get, $\frac{r/5}{r/15}=\frac{R+10}{R-10}$

$$3=\frac{R+10}{R-10}$$

3R – 30 = R+10
R = 20

28. Let p, q be prime numbers such that $n^{3pq}-n$ is a multiple of 3pq for all positive integers n. Find the least possible value of p + q.

Sol. Using Euler's Totient function $a^{\phi(n)}\equiv 1(\text{mod } n)$

$3pq|n^{3pq}-n \Rightarrow 3pq|n(n^{3pq-1}-1)$

it is easy to see that p,q have got to be both odd and neither can be 3

$\therefore$ 3, p, q are pair wise relatively prime $\Rightarrow \phi(3pq) = 2(p-1)(q-1)$

$\therefore$ $3pq|n(n^{3pq-1}-1)\forall n\in \mathbb{N}$ if $\phi(3pq)|(3pq-1)$

$\Leftrightarrow 2(p-1)(q-1) \mid (3pq-1)$

since p.q are odd $\Rightarrow\ 2 \mid (3pq-1)$

$\left.\begin{array}{l}(p-1) \mid (3pq-1) \Leftrightarrow (p-1) \mid (3q-1) \\ (q-1) \mid (3pq-1) \Leftrightarrow (q-1) \mid (3p-1)\end{array}\right\}$ Both should hold simultaneously

the least p+q is 11 + 17 = 28 (for p = 11, q = 17)

29. For each positive integer n, consider the highest common factor h_n of the two numbers n! + 1 and (n + 1)!. For n < 100, find the largest value of h_n.

Sol. $\lfloor\underline{n}+1$ will be divisible by (n+1) if (n+1) is a prime number (by Wilson Theorem)

so, H.C.F of ($\lfloor\underline{n}+1$, $\lfloor\underline{n+1}$) = $(n+1)$ if (n + 1) is prime

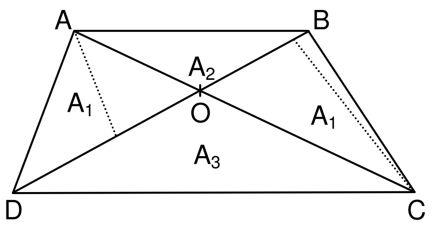
H.C.F of $(\lfloor\underline{n}+1, \lfloor\underline{n+1}) = 1$ if (n+1) is not prime

so, only possibility for maximum H.C.F when n = 96

for n = 96

Both $\lfloor\underline{96}+1$ and $\lfloor\underline{97}$ will be divisible by 97

30. Consider the areas of the four triangles obtained by drawing the diagonals AC and BD of a trapezium ABCD. The product of these areas, taken two at a time, are computed. If among the six products so obtained, two products are1296 and 576, determine the square root of the maximum possible area of the trapezium to the nearest integer.



Sol. Clearly $\dfrac{A_1}{A_2} = \dfrac{A_3}{A_1} \Rightarrow A_1^2 = A_2A_3$

Now 6 products are $A_1^2, A_1^2, A_1A_2, A_1A_2, A_1A_3, A_1A_3$

reduces to 3 cases only

Case-I: When $A_1^2 = 1296$ and $A_1A_2 = 576$ gives

$2A_1 + A_2 + A_3 = 169$

Case-II: When $A_1^2 = 576$ and $A_1A_2 = 1296$ gives

$2A_1 + A_2 + A_3 = 112.66 < 169$

Case-III: When $A_1A_2 = 1296$ and $A_1A_3 = 576$ gives

$2A_1 + A_2 + A_3 = 25\sqrt{24} < 169$

Hence, max value of $\sqrt{2A_1 + A_2 + A_3} = 13$